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ANALYSIS OF GENERAL STRUCTURAL NETWORKS BY MATRIX METHODS

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SUMMARY

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In this report, an analysis for a complicated Structural System is presented, using the concepts and analogies of the corresponding electrical system. Any complex system, electrical or mechanical, can be broken down into a series of simple systems and connected together, not violating the equilibrium or compatibility conditions. Transfer Matrix analysis is a computer oriented convenient method for complex systems.

Author

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LIST OF SYMBOLS

A, B, C, D	Subsystem element
A, B	Transfer matrix
A_{ij}, B_{ij}	Element in a matrix
C	Product matrix of B and A
D	Displacement, rotation vector
F, G	Force, applied force
i, j	Subscript
K	Spring constant
K_{ij}	Element in a matrix
m, n	Row, column
m, M	Mass
M, N	Moment, applied moment
P	Applied force, moment vector
T	Overall transfer matrix
U, V	Translations component
X	Deflection

GREEK ALPHABET

θ, ψ	Rotation component
ω	Angular frequency
ω_0	Angular resonant frequency

ANALYSIS OF GENERAL STRUCTURAL NETWORKS BY MATRIX METHODS

In order to apply matrix techniques, such as the transfer matrix method, to the analysis of complex structural systems, it is necessary to break up, or divide, the structure into subsystems or components, each of which can be analyzed separately. The load-deflection characteristics of the entire structure, whether statically or dynamically loaded, are then found by mathematically connecting all of the known load - deflection characteristics (or solutions) of the individual subsystems in a manner which is consistent with the physical connections between these subsystems. In this process of reconnecting the subsystems, it is apparent that, in effect, the entire complex structure has been replaced by an equivalent mechanical network, or circuit, the elements of which are the subsystems. The networks may be simple or complex depending upon the complexity of the entire structure and the number of subsystems into which this structure is divided.

Mechanical network concepts have been used extensively to analyze elementary structural components such as lumped spring-mass systems and continuous beam, ring, and plate structures which are represented by equivalent lumped spring-mass systems. The primary advantage of the mechanical network lies in its analogy to electrical networks which can be analyzed with relative ease and for which laws and systematic methods of attack have been thoroughly investigated and documented in the literature. In particular, electrical circuits can be analyzed by matrix methods through the use of Kirchoff's laws concerning the voltages and currents at electrical circuit junctions. Similar matrix methods can be applied to mechanical networks with the understanding, however, that Kirchoff's laws must be generalized to account for the multi-coordinate deflection and load properties of mechanical junctions.

In an electrical circuit, the voltages at a given junction must be the same for all electrical branches connected to that junction. Analogously, the deflection in a given direction, or rotation about a given axis, at the junction of a mechanical network must be the same for all mechanical branches connected to that junction. However, in the electrical circuit, voltage at a junction is a simple scalar quantity; whereas, in a mechanical circuit, deflection and rotation at a junction are vector quantities each having three spatial components. It is necessary to generalize one of Kirchoffs electrical laws by stating that at a mechanical junction, the vector displacements and the vector rotations must be the same for all mechanical branches connected to that junction. This generalization is clearly based on the compatibility requirements for physical systems.

Based on the equilibrium requirements for physical systems, an analogous generalization can be made with regard to the net loads acting at a mechanical junction. In the electrical circuit, the sum of all the signed branch currents must be equal to zero, where current is also a scalar (algebraic) quantity. For a mechanical junction the sum of all the forces in a given direction and the sum of all the moments about a given axis must be equal to zero. The forces and moments at a mechanical junction are vector quantities each having three spatial components. Thus, the generalization of Kirchoff's second law to mechanical circuits requires that the sum of all of the branch vector forces and vector moments at a mechanical junction be equal to zero.

With these simple rules, any mechanical network can be described mathematically in a matrix form and so analyzed. In general, the structure may be redundant or nonredundant, statically or dynamically loaded, or free from applied loads in the dynamic case. To illustrate the application of matrix methods to mechanical networks, several examples of a rather general nature are discussed below. In the following examples, it is assumed for simplicity that all of the structures are in a steady vibration state at some frequency ω.

Consider first the simple two element mechanical system, shown in Figure 1 below, which consists of a spring and mass in series. The quantities x_1 and x_2 denote deflection amplitudes at frequency ω and F_1 and F_2 denote applied force amplitudes at frequency ω. This structural

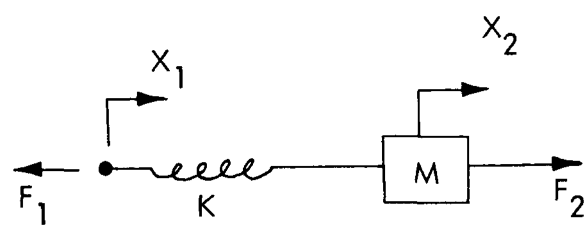


Figure 1: Simple two-element series type structure

system is now divided into two subsystems as shown in Figure 2 below. The junction deflections and loads shown in Figure 2 were chosen so as to automatically satisfy the required compatability and equilibrium conditions. From elementary physics, the transfer matrices

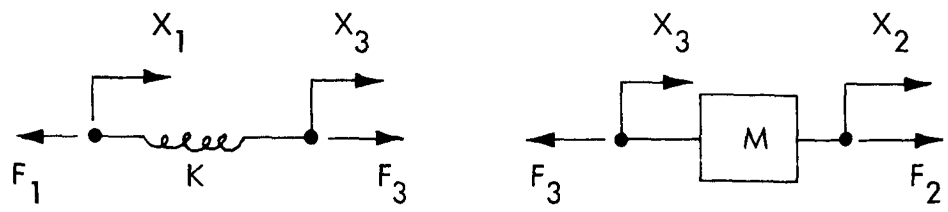


Figure 2: Segment two element series type structure

for these two subsystems are

$$\begin{vmatrix} X_3 \\ F_3 \end{vmatrix} = \begin{vmatrix} 1 & 1/K \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ F_1 \end{vmatrix}$$

$$\begin{vmatrix} X_2 \\ F_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{vmatrix} \cdot \begin{vmatrix} X_3 \\ F_3 \end{vmatrix}$$

Combining these two equations gives the transfer matrix across the entire system:

$$\begin{vmatrix} X_2 \\ F_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1/K \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ F_1 \end{vmatrix}$$

or

$$\begin{vmatrix} X_2 \\ F_2 \end{vmatrix} = \begin{vmatrix} 1 & 1/K \\ -m\omega^2 & 1-(\omega/\omega_o)^2 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ F_1 \end{vmatrix} \qquad \text{Transfer Matrix Equation} \qquad (1)$$

where ω_o denotes the system resonant frequency.

$$\omega_o^2 = K/m.$$

The 2 x 2 matrix in (1) is called the system transfer matrix. The system flexibility matrix equation can be obtained from (1) by solving for X_1 and X_2 in terms of F_1 and F_2. The result is:

$$\begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \frac{1}{m\omega^2} \begin{vmatrix} 1-(\omega/\omega_o)^2 & -1 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} F_1 \\ F_2 \end{vmatrix} \qquad \text{Flexibility Matrix Equation} \qquad (2)$$

The square matrix in (2) along with the factor $(1/m\omega^2)$ is called the flexibility matrix.

It is to be noted that the flexibility matrix does not exist if the system is statically loaded; i.e. if $\omega = 0$. This is generally true for all structures unless the structures are so constrained that neither rigid body translations nor rigid body rotations can occur. Where there are no permissible rigid body motions permissible, the flexibility matrix will exist in general.

The stiffness matrix equation can be obtained from either (1) or (2) by solving for the loads F_1 and F_2 in terms of the deflections X_1 and X_2. The result is as follows:

$$\begin{vmatrix} F_1 \\ \\ F_2 \end{vmatrix} = K \begin{vmatrix} -1 & 1 \\ \\ -1 & 1-(\omega/\omega_o)^2 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ \\ X_2 \end{vmatrix} \quad \text{Stiffness Matrix Equation} \qquad (3)$$

The 2 x 2 matrix in (3), along with the factor K, is called the stiffness matrix of the system.

The above elementary example was presented in order to show, in simple terms, the transfer matrix process, and to define the transfer, flexibility, and stiffness matrices. With this brief introduction to the basic concepts, it is now possible to consider a much more complex three-dimensional structural system such as that shown in Figure 3. This system consists of two principal subsystems attached at the point 2; the one system having an applied force vector $\overline{F}$ and moment vector $\overline{M}$ at point 1, and the other system having an applied force vector $\overline{G}$ and moment vector $\overline{N}$ at point 3. This system can be more conveniently represented

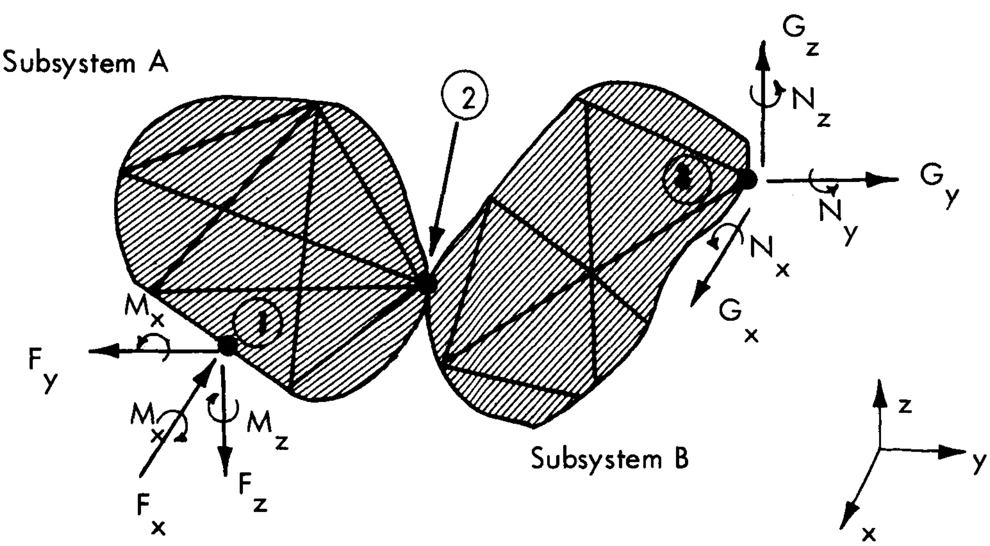


Figure 3: Complex structure with two distinguishable subsystems attached at point 2.

now by the series circuit shown in Figure 4. The quantities D_1 and D_3 represent the displacement and rotation vectors at points 1 and 3 respectively, while the quantities P_1 and

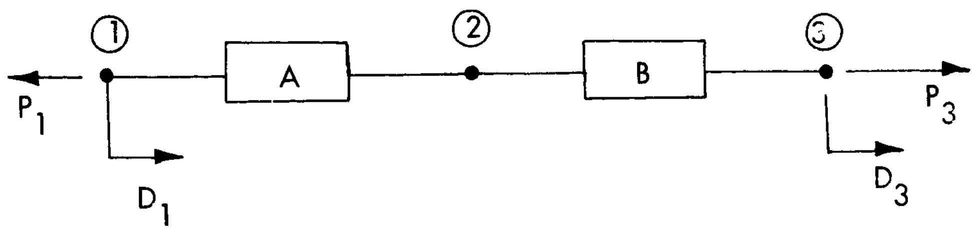


Figure 4: Network for two subsystems connected in series.

P_3 represent the applied force and moment vectors at points 1 and 3 respectively. Represented as column matrices, the quantities are

$$D_1 = \begin{vmatrix} U_x \\ U_y \\ U_z \\ \theta_x \\ \theta_y \\ \theta_z \end{vmatrix} \qquad D_3 = \begin{vmatrix} V_x \\ V_y \\ V_z \\ \psi_x \\ \psi_y \\ \psi_z \end{vmatrix}$$

$$P_1 = \begin{vmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{vmatrix} \qquad P_3 = \begin{vmatrix} G_x \\ G_y \\ G_z \\ N_x \\ N_y \\ N_z \end{vmatrix}$$

where

U_x, U_y, U_z = translation components at point 1
$\theta_x, \theta_y, \theta_z$ = rotation components at point 1
V_x, V_y, V_z = translation components at point 3
ψ_x, ψ_y, ψ_z = rotation components at point 3

The network in Figure 4 can now be cut as shown in Figure 5, with D_2 representing the generalized junction deflection and P_2 representing the generalized junction loads. The sign conventions for the loads and deflections at the junction were chosen so that equilibrium

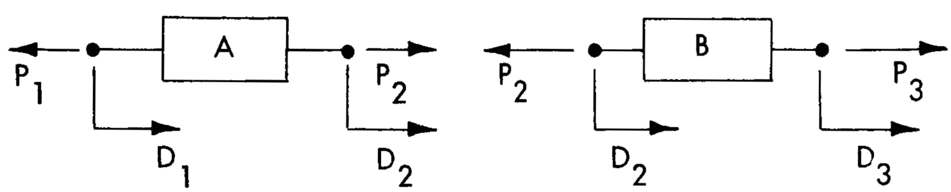


Figure 5: Segmented series network with intermediate junction loads and deflections which satisfy the equilibrium and compatibility requirements.

and compatibility requirements of the physical system were met.

If A and B denote the transfer matrices for the two subsystems, then the transfer matrix equations for the two subsystems are

$$\begin{vmatrix} D_2 \\ P_2 \end{vmatrix} = A \cdot \begin{vmatrix} D_1 \\ P_1 \end{vmatrix}$$

$$\begin{vmatrix} D_3 \\ P_3 \end{vmatrix} = B \cdot \begin{vmatrix} D_2 \\ P_2 \end{vmatrix}$$

combining these equations gives the general transfer matrix for the entire structure, namely

$$\begin{vmatrix} D_3 \\ P_3 \end{vmatrix} = C \cdot \begin{vmatrix} D_1 \\ P_1 \end{vmatrix} \qquad \text{General Transfer Matrix Equation} \qquad (4)$$

where

$$C = B\,A$$

It is seen that the general form of (4) is similar to that of (1). Hence, following the definition given for the flexibility and stiffness matrix equations for the elementary system in Figure 1, comparable equations can be developed immediately from (4) for the complex system shown in Figure 3. To do this, write (4) in the following form

$$\begin{vmatrix} D_3 \\ P_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 \\ C_3 & C_4 \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ P_1 \end{vmatrix}$$

from which it follows that

$$D_3 = C_1 D_1 + C_2 P_1 \tag{5}$$

$$P_3 = C_3 D_1 + C_4 P_1 \tag{6}$$

Solving (6) for D_1,

$$D_1 = C_3^{-1} P_3 - C_3^{-1} C_4 P_1 \tag{7}$$

Substituting (7) into (6), gives

$$D_3 = (C_2 - C_1 C_3^{-1} C_4) P_1 + C_1 C_3^{-1} P_3 \tag{8}$$

From (7) and (8), the flexibility matrix equation becomes

$$\begin{vmatrix} D_1 \\ \\ D_3 \end{vmatrix} = \left|\begin{array}{c:c} -C_3^{-1} C_4 & C_3^{-1} \\ & \\ C_2-C_1C_3^{-1}C_4 & C_1 C_3^{-1} \end{array}\right| \cdot \begin{vmatrix} P_1 \\ \\ P_3 \end{vmatrix} \quad \text{Flexibility Matrix Equation} \tag{9}$$

Solving (5) for P_1,

$$P_1 = C_2^{-1} D_3 - C_2^{-1} C_1 D_1 \tag{10}$$

Substituting (10) into (6) and solving for P_3 ,

$$P_3 = (C_3 - C_4 C_2^{-1} C_1) D_1 + C_4 C_2^{-1} D_3 \tag{11}$$

From (10) and (11) the general stiffness matrix equation becomes,

$$\begin{vmatrix} P_1 \\ \\ P_3 \end{vmatrix} = \begin{vmatrix} -C_2^{-1} C_1 & C_2^{-1} \\ \\ C_3 - C_4 C_2^{-1} C_1 & C_4 C_2^{-1} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ \\ D_3 \end{vmatrix} \qquad \text{Stiffness Matrix Equation} \qquad (12)$$

It is to be noted that the flexibility matrix in (9) will not exist for a statically loaded structure, where $\omega = 0$, unless the structure is constrained against rigid body motions. In general though, the flexibility matrix will exist for a non-zero excitation frequency.

The structural system in Figure 3 can be generalized to a number of series - connected subsystems such as shown in Figure 6. The equivalent network for the structure in Figure 6 is shown in Figure 7. If A, B, C, D represent the transfer matrices for the four subsystems,

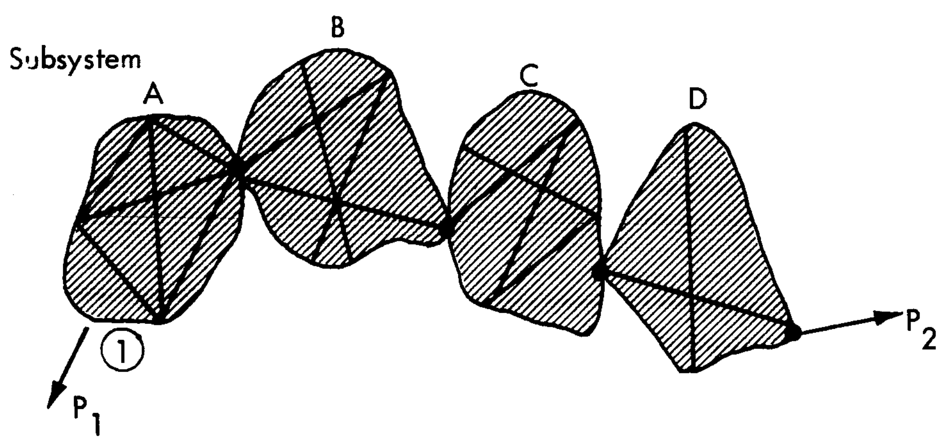


Figure 6: Series connected structural subsystems.

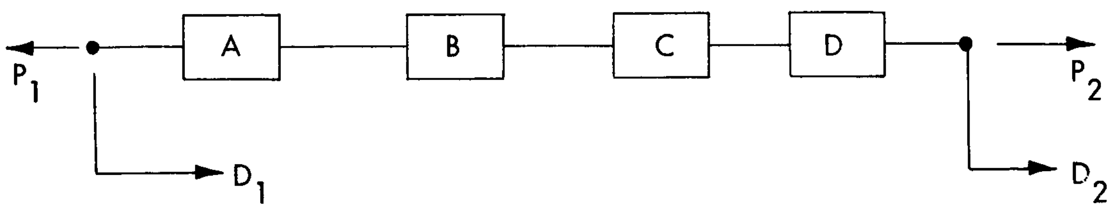


Figure 7: Series mechanical network for the four subsystems structure of Figure 6.

then the transfer matrix from the point 1 to the point 2 is given by the equation

$$\begin{vmatrix} D_2 \\ P_2 \end{vmatrix} = D\ C\ B\ A \begin{vmatrix} D_1 \\ P_1 \end{vmatrix}$$

where

P_1 = generalized load applied at point 1.

P_2 = generalized load applied at point 2.

D_1 = generalized deflection at point 1.

D_2 = generalized deflection at point 2.

Further generalizations of the structural systems in Figure 6 to n series connected substructures are now obvious.

Consider now another type of structure in which there exist two distinct parallel load paths, such as is shown in Figure 8. As before, the deflections D_1 and D_2 and the loads P_1 and

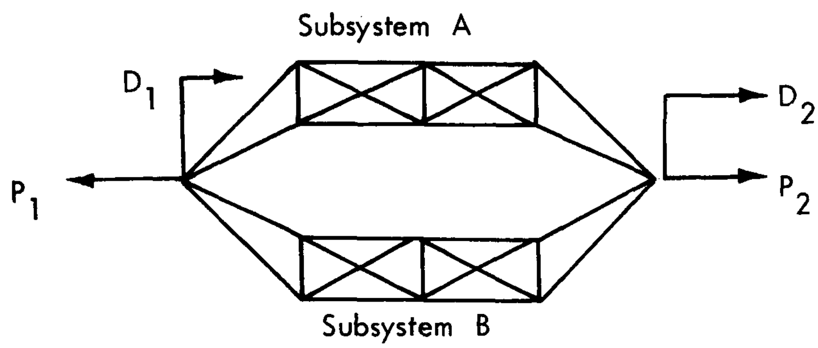


Figure 8: Complex structure with parallel subsystems

P_2 are generalized so that each quantity contains six components. In Figure 9, this structure is shown separated into two parts along with the load components for each branch at each junction. The deflections clearly satisfy the conditions of compatibility. The equilibrium requirements at each junction are satisfied by the following loads equations

$$
\begin{aligned}
P_1 &= P_1' + P_1'' \\
P_2 &= P_2' + P_2''
\end{aligned}
\qquad (13)
$$

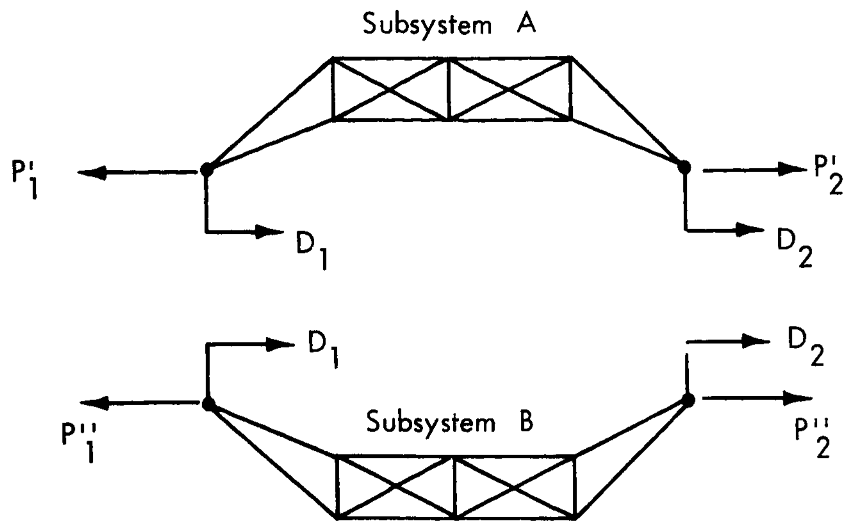


Figure 9: Complex structure branch load components and deflections of parallel subsystems.

If A and B denote the transfer matrices for the two subsystems, then the transfer matrix equations are:

$$\begin{vmatrix} D_2 \\ P'_2 \end{vmatrix} = A \begin{vmatrix} D_1 \\ P'_1 \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ P'_1 \end{vmatrix} \tag{14}$$

$$\begin{vmatrix} D_2 \\ P''_2 \end{vmatrix} = B \begin{vmatrix} D_1 \\ P''_1 \end{vmatrix} = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ P''_1 \end{vmatrix} \tag{15}$$

Equations (13), (14) and (15) represent six equations in eight unknowns so that six of the unknowns can be expressed in terms of the other two which act as independent variables. Solving for D_2 and P_2 in terms of D_1 and P_1 gives the following transfer matrix equation

$$\begin{vmatrix} D_2 \\ P_2 \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ P_1 \end{vmatrix} \tag{16}$$

where

$$\begin{aligned}
T_{11} &= \left[I + A_{12} B_{12}^{-1} \right]^{-1} \left[A_{11} + A_{12} B_{12}^{-1} B_{11} \right] \\
T_{12} &= \left[I + A_{12} B_{12}^{-1} \right]^{-1} A_{12} \\
T_{21} &= A_{21} + B_{21} + (A_{22} - B_{22}) B_{12}^{-1} B_{11} \\
&\qquad + (B_{22} - A_{22}) B_{12}^{-1} \left[I + A_{12} B_{12}^{-1} \right]^{-1} \left[A_{11} + A_{12} B_{12}^{-1} B_{11} \right] \\
T_{22} &= A_{22} + (B_{22} - A_{22}) B_{12}^{-1} \left[I + A_{12} B_{12}^{-1} \right]^{-1} A_{12}
\end{aligned} \tag{17}$$

Solving for P_1 and P_2 in terms of D_1 and D_2 gives the following stiffness matrix equation

$$\begin{vmatrix} P_1 \\ P_2 \end{vmatrix} = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ D_2 \end{vmatrix} \tag{18}$$

where

$$
\begin{aligned}
K_{11} &= -A_{12}^{-1} A_{11} - B_{12}^{-1} B_{11} \\
K_{12} &= A_{12}^{-1} + B_{12}^{-1} \\
K_{21} &= A_{21} + B_{21} - A_{22} A_{12}^{-1} A_{11} - B_{22} B_{12}^{-1} B_{11} \\
K_{22} &= A_{22} A_{12}^{-1} + B_{22} B_{12}^{-1}
\end{aligned}
\tag{18}
$$

The key feature of the structures in Figures 3 and 6 is that two adjacent subsystems are attached at a single point. Figure 10 shows a slightly more general case than Figure 3 in the sense that the two distinguishable subsystems are attached at two points. These two subsystems are shown separately in Figure 11 along with the two intermediate junction

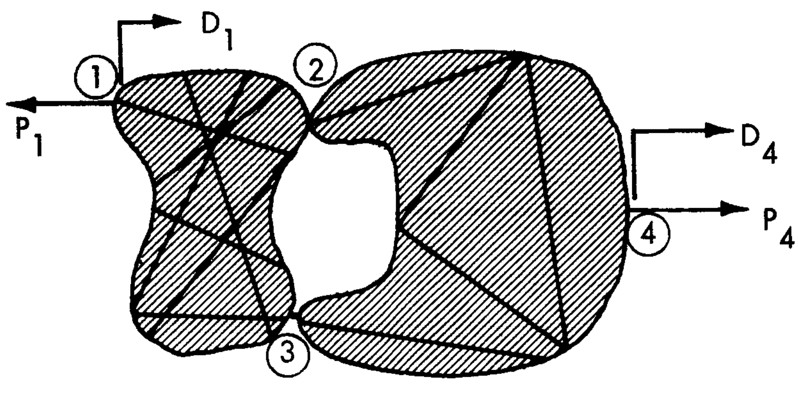


Figure 10: Complex structure with two distinguishable subsystems attached at two points.

deflections and loads in generalized form. The junction loads and deflections were chosen so as to satisfy the equilibrium and compatability requirement.

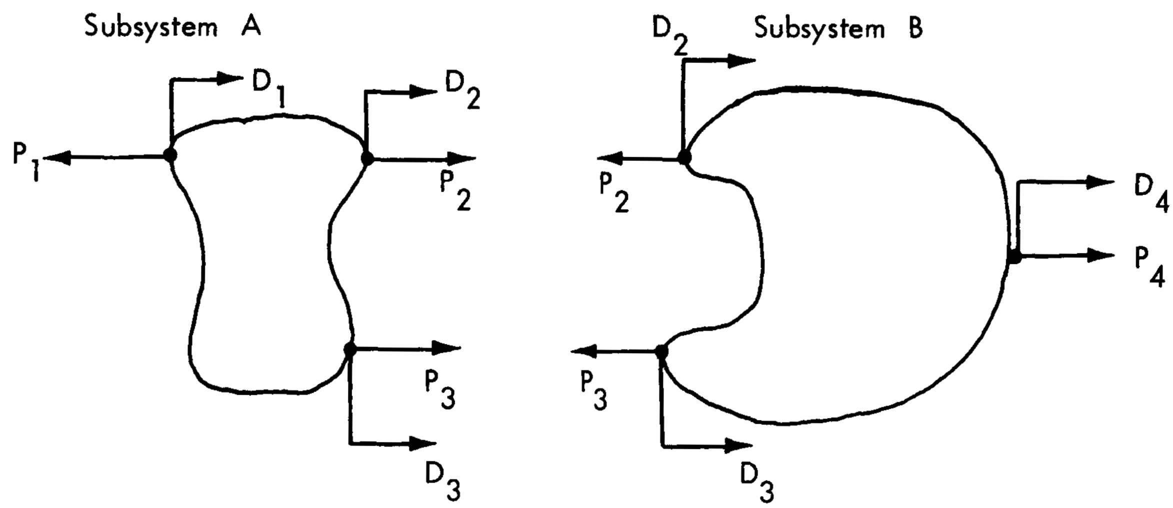


Figure 11: Diagram of two subsystems showing loads and deflections at intermediate junctions.

From Figure 11 it is clear that there are two parallel paths by which the loads can be transferred from point 1 to point 4. In this sense, the structure shown in Figure 10 resembles that of Figure 8. The essential difference is that the two load paths within the individual subsystems in Figure 10 are indistinguishable and in general they will overlap, or have a common part. The transfer matrix for the complete structure can be obtained without difficulty, however it is possible to determine the overall stiffness, matrix and transfer matrix for a much more general use with little additional effort.

Consider a complex structural system which has two distinguishable subsystems, such as shown in Figure 12 ; and assume that there are m generalized applied loads acting on each subsystem and n junctions between the two subsystems. It is assumed that each

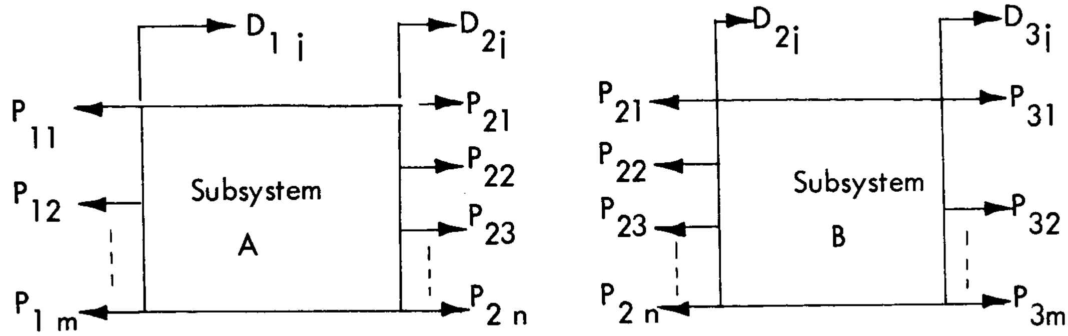


Figure 12: Two subsystems with n contact points 2nd m applied loads on each subsystem.

of the two subsystems is analyzable, and that the stiffness matrices for each subsystem are known. The stiffness matrix equation for each subsystem can be written in the form

$$\begin{array}{c} m \\ n \end{array} \begin{bmatrix} P_{1i} \\ P_{2i} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ \; m \quad\quad n \end{array} \cdot \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} \begin{array}{c} m \\ n \end{array} \qquad (19)$$

$$\begin{bmatrix} P_{2i} \\ P_{3i} \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \cdot \begin{bmatrix} D_{2i} \\ D_{3i} \end{bmatrix} \tag{20}$$

Thus, from (19) and (20),

$$P_{1i} = A\, D_{1i} + B\, D_{2i} \tag{21}$$

$$P_{2i} = C\, D_{1i} + D\, D_{2i} \tag{22}$$

$$P_{2i} = A'\, D_{2i} + B'\, D_{2i} \tag{23}$$

$$P_{3i} = C'\, D_{2i} + D'\, D_{3i} \tag{24}$$

Subtracting (23) from (22) gives

$$(D - A')\, D_{2i} + C\, D_{1i}\ B'\, D_{3i} = 0 \tag{25}$$

or

$$D_{2i} = (D - A')^{-1} B' D_{3i} - (D - A')^{-1} D D_{1i} \tag{26}$$

Substituting (26) into (21) and (24), and writing the result in matrix form, gives the overall stiffness matrix equation for the structure:

$$\begin{bmatrix} P_{1i} \\ P_{3i} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \cdot \begin{bmatrix} D_{1i} \\ D_{3i} \end{bmatrix} \tag{27}$$

where

$$\begin{aligned} K_{11} &= A - B (D - A')^{-1} C \\ K_{12} &= B (D - A')^{-1} B' \\ K_{21} &= - C' (D - A')^{-1} C \\ K_{22} &= C' (D - A')^{-1} B' + D' \end{aligned} \tag{28}$$

The transfer matrix can be obtained from (27) and is

$$\begin{bmatrix} D_{3i} \\ P_{3i} \end{bmatrix} = \begin{bmatrix} -K_{12}^{-1} K_{11} & K_{12}^{-1} \\ K_{21} - K_{22} K_{12}^{-1} K_{11} & K_{22} K_{12}^{-1} \end{bmatrix} \cdot \begin{bmatrix} D_{1i} \\ P_{1i} \end{bmatrix} \tag{29}$$

It is interesting to note that the two subsystems shown in Figure 12 can be thought of as being in series, in a generalized sense, just like the subsystems shown in Figure 3. The transfer matrix equation (29) is simply a generalization of the analogous equations for subsystems which have single junctions with adjacent subsystems. Similarly, multijunction subsystems can appear in parallel type networks such as those shown in Figure 13 below.

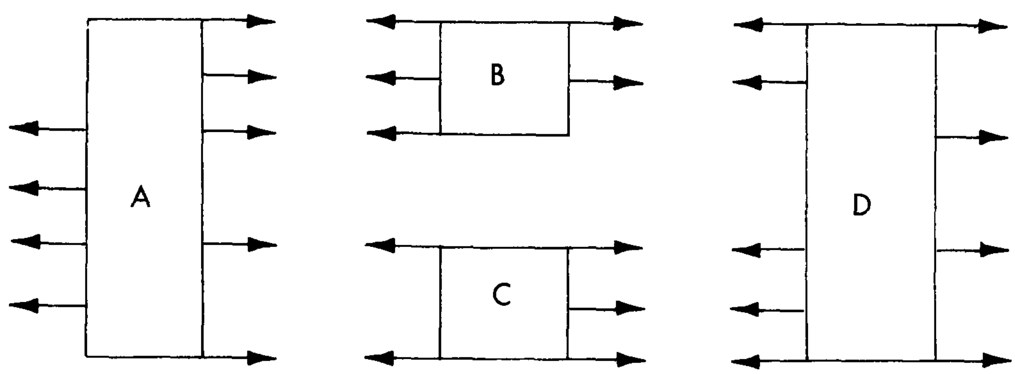


Figure 13: Parallel arrangement of multijunction subsystems.

However, the arrangement shown in Figure 13 cannot be treated in exactly the same manner as the parallel network of Figure 8. The transfer matrix for this system can be obtained from the stiffness, flexibility or transfer matrices of each of the individual subsystems.

In the above discussion, it was demonstrated how subsystems can be recombined into mechanical networks which mathematically simulate the physical interconnections of the structural subsystems. Both single point interconnections and multijunction subsystems were considered. It is of interest now to consider the network structure of typical subsystems and how these networks can be analyzed.

Large complex vehicle structures such as aircraft, aerospace vehicles, space stations, etc., are often composed of several different types of structures such as wings, tail assemblies with fins, engine and engine mounts, fuselages, cylindrical fuel tanks, solid rocket engines, large inner fuel tanks, instrument bays, space capsules, etc. When segmenting the entire structure for analysis, it is generally convenient to divide the structure into its major structural components, like those just listed, and then treat these components as sub-assemblies of the type referred to in the previous discussion. Each of these components can then be analyzed separately with the advantage that appropriate methods of analysis can be applied to each different type of subsystem. When this is done, it is clear that the subsystems themselves will generally be complex structures, the analysis of which will usually require a finite element approach.

The primary advantage of the finite element approach to structural analysis lies in the ability of the investigator to divide a complex structure into a large number of small structural elements each of which can be analyzed using basic strength of materials and dynamics concepts. Thus, the investigator would prefer to use beam segments, flexures, lumped masses, springs, dashpots, ring segments and even simple plates and membranes as the basic building blocks for representing a complex structure. As a result, it is likely that the above major structural subsystems will be represented by networks of these elemental types of structures.

Minimum weight requirements for flight vehicles and other aerospace vehicles dictate that the majority of the structures must consist of an orthogonal framework of stiffeners overlayed with thin skins or light honeycomb skins. Structures of this type will usually be represented by two dimension networks which are fairly regular, having repeative patterns of certain basic type circuits. Exceptions to this are high load carrying aerodynamic surfaces such as wings, fins, etc. which for sake of accuracy must be represented by three dimensional networks. Within a given subsystem, such essentially two dimensional structures may be joined to form three dimensional structures such as a bulkhead mounted within a cylindrical fuselage shell. However, the essential two dimensionality of much of the vehicle structure certainly exists.

Typically, these stiffened curved shells and flat plates, or the framework itself, will be represented by orthogonal and bridge networks of the type shown in Figure 14. The individual elements in these networks will often consist of masses, flexures, rods, beams, etc. The subsystems are therefore expected to consist of networks of the type shown in Figure 14, and any one subsystem may consist of several such networks interconnected in three dimensional box-like arrays. Such systems are readily evaluated by transfer matrix methods.

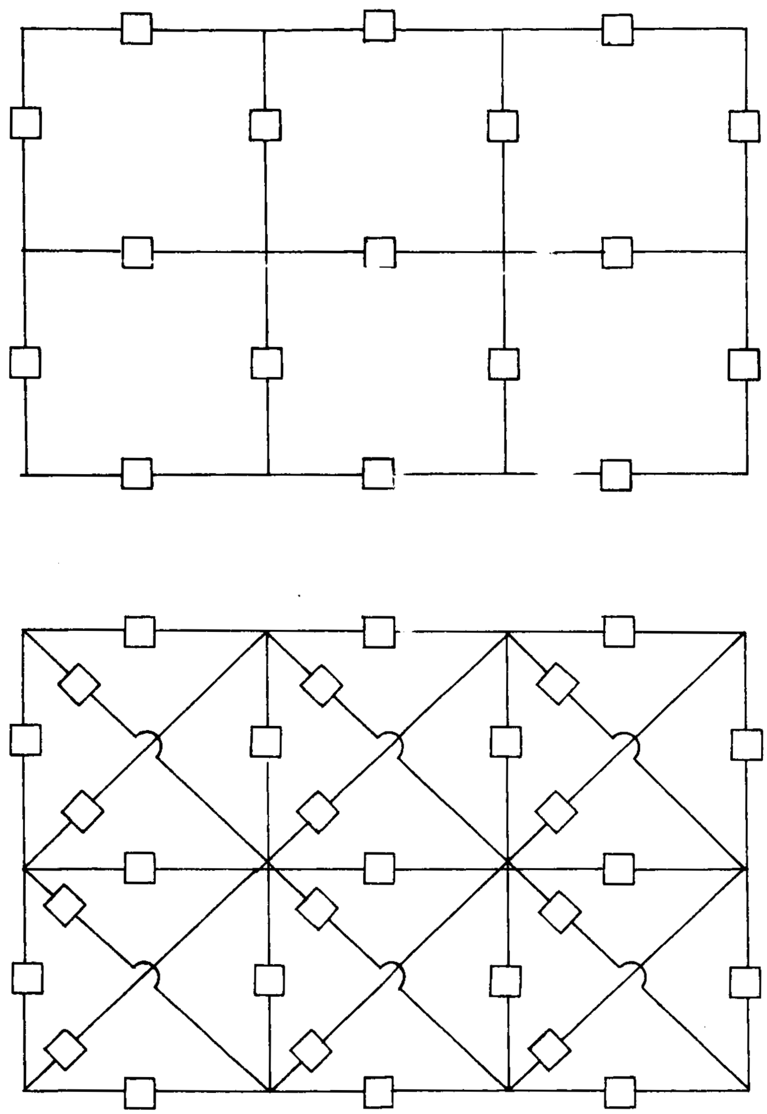


Figure 14: Typical orthogonal and bridge type mechanical mechanical networks of two dimensional structure.

In order to analyze a network like the bridge circuit in Figure 14, it is necessary to segment the network into fields and rows as is shown in Figure 15. Stiffness matrices are then computed for the rows and for the fields; and the transfer matrices are determined from these stiffness matrices by the matrix manipulation scheme derived earlier in this section.

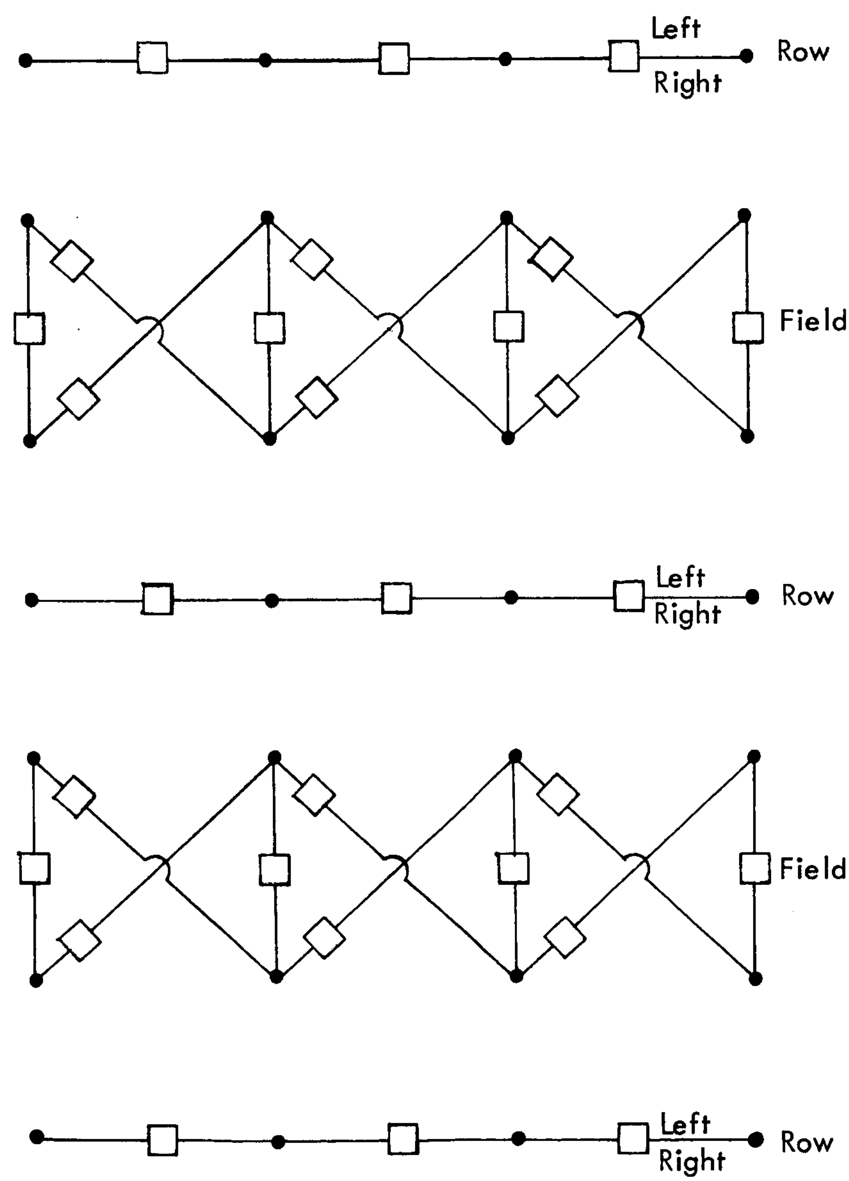


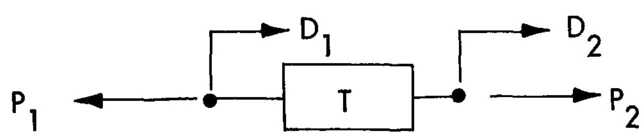
Figure 15: Segmented network structure

The transfer matrix for a row gives the deflections and loads on the right side of an entire row in terms of the deflections and loads on the left side of an entire row. Similarly for the transfer matrix of a field of vertical and diagonal branches, these concepts are more precisely illustrated in Wyle Laboratories - Research Staff report WR 65-29, in which the transfer matrices are developed in detail for a uniform rectangular plate using the Hrennikoff equivalent framework model for the plate .

In the analysis of two dimensional structures, it is often convenient to develop a stiffness matrix which gives the loads around the entire edge of a shell or plate in terms of the deflections of that edge. This is a useful procedure for elastically and inertially coupling a plate to stiffeners along the edges of the plate, and for performing a complete and accurate dynamic analysis of stiffened plate structures. This problem is discussed in WR 65-29 and it is shown in that discussion that it is necessary to perform two transfer matrix analyses along the two principal directions of the plate, and to combine the two rectangular plate transfer matrices into a single square stiffness matrix.

Many other networks which are combinations of those discussed or which are distinctly different from those shown above can exist, including all of those found in electrical engineering such as bridge circuits. No attempt is made here to discuss all of the different types of circuits. It has been demonstrated however, that systematic matrix manipulations can be applied to these networks which will pemit the analysis of very complex structures.

As a final comment, it is interesting to note how equations for systems resonant frequencies can be obtained. Let T denote the overall transfer matrix for the elastic structural system shown in Figure 14



so that

$$\begin{vmatrix} D_2 \\ P_2 \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \cdot \begin{vmatrix} D_1 \\ P_1 \end{vmatrix}$$

If both ends of the structure are free, then $P_1 = P_2 = 0$ from which it follows that

$$\det T_{21} = 0 \qquad \text{both ends free}$$

This determinant equation is a function of the frequency ω which appears in the elements of T_{21}, and hence there will be certain discrete values of ω which satisfy this equation. If both ends are fixed then $D_1 = D_2 = 0$ from which it follows that

$$\det T_{12} = 0 \qquad \text{both ends fixed}$$

If point 1 is fixed and 2 is free, then $D_1 = P_2 = 0$ for which

$$\det T_{22} = 0 \qquad \begin{matrix}\text{Point 1 fixed} \\ \text{Point 2 fixed}\end{matrix}$$

If point 2 is fixed and 1 is free, then $D_2 = P_1 = 0$, for which

$$\det T_{11} = 0 \qquad \begin{matrix}\text{point 2 fixed} \\ \text{point 1 free}\end{matrix}$$

Other, more complex boundary conditions are possible in which one end of the system may be fixed with respect to certain displacements while other displacements are unconstrained. In such cases, the determinant of some submatrix within the transfer matrix will be equal to zero; and this submatrix may include portions of all four of the T_{ij} matrices.